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EXPLICIT EVALUATION OF SOME OF THE THETA-FUNCTION IDENTITIES

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Abstract. In this paper, we evaluate two parameters $h_{k,n}$ and $h'_{k,n}$ of some P-Q type Theta functions $\varphi(q)$ for any positive real numbers k and n. During this process, we also evaluate Ramanujan's cubic continued fraction.

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1. INTRODUCTION

For any complex number z, $q = e^{2\pi iz}$, $Im(z) > 0$, define

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} =: \Theta_3(0, 2z)$$

and

$$f(-q) := (q;q)_\infty = q^{-1/24}\eta(z),$$

where Θ_3 is the classical theta-function [23, p. 464] and $\eta(z)$ denotes the Dedekind eta-function and $(a;q)_\infty$ is defined by

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

It is precisely assumed in the sequel that $|q| < 1$ always. Recently, J. Yi [24] evaluated many new values of $\varphi(q)$ using modular identities, transformation formula for theta-functions and are defined as follows:

Definition 1. For any positive real number k and n we have

$$h_{k,n} := \frac{\varphi(q)}{k^{1/4}\varphi(q^k)} = \frac{\Theta_3(0, i\sqrt{n/k})}{k^{1/4}\Theta_3(0, i\sqrt{nk})} \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.1}$$

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$$h'_{k,n} := \frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)} = \frac{\Theta_3(0,1+2i\sqrt{n/k})}{k^{1/4}\Theta_3(0,1+2i\sqrt{nk})} \qquad q = e^{-2\pi\sqrt{n/k}}. \tag{1.2}$$

Also it is observed that

i. $h_{k,1} = 1$,
ii. $h_{k,1/n} = h_{k,n}^{-1}$,
iii. $h_{k,n} = h_{n,k}$.

The Ramanujan-Göllnitz-Gordan continued fraction $H(q)$ is defined as

$$H(q) := \frac{q^{1/2}}{1+q}{}_{+}\frac{q^2}{1+q^3}{}_{+}\frac{q^4}{1+q^5}{}_{+\dots}.$$

The above continued fraction was introduced by S. Ramanujan in his second notebook [16, p. 229]. H. Göllnitz [11] and B. Gordon [12] rediscovered $H(q)$ without knowing of Ramanujan's work. Ramanujan also recorded following two identities for $H(q)$ in his second notebook [16, p. 229],

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \quad \text{and} \quad \frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}.$$

Proofs of the above two identities can be found in [4, p. 221]. H. H. Chan and S. S. Haung [10], have established many identities for $H(q)$, which are analogues to the results of famous Roger-Ramanujan continued fraction and Ramanujan's cubic continued fraction. Chan and Haung [10] have also derived some explicit formulas for evaluating $H(e^{-\pi\sqrt{n}/2})$ in terms of Ramanujan-Weber class invariants. Recently C. Adiga et. al. [2] have established several modular relations for the Rogers-Ramanujan type functions of order eleven which are analogues to Ramanujan's forty identities and also they established certain interesting partition theoretic interpretations. H. M. Srivastava and M. P. Chaudhary [17] established a set of four new results which depict the interrelationships between q-identities, continued fraction identities and combinatorial partition identities. Also in [18, 19] H. M. Srivastava et. al. deduced some q-identities involving theta functions. These q-identities have relationship among three of the theta-type functions which arise from Jacobi's triple product identity. In [22], K. R. Vasuki and B. R. Srivatsa Kumar proved the following:

Lemma 1. *For $q = e^{-\pi\sqrt{k/2}}$, let*

$$J_k := \sqrt{2}\frac{\varphi^2(q^2)}{\varphi^2(q)},$$

then

i. $J_k J_{1/k} = 1$,
ii. $J_1 = 1$,
iii. $H(q) = \sqrt{\dfrac{\sqrt[4]{2} - \sqrt{J_k}}{\sqrt[4]{2} + \sqrt{J_k}}}$.

In his first notebook [16] Ramanujan recorded many elementary values of $\varphi(q)$. Particularly, he recorded $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$ for $n = 1,2,4,8,1/2$ and $1/4$. All these values are proved by Berndt [5, p. 325]. Ramanujan also recorded non-elementary values of $\varphi(e^{-n\pi})$ for $n = 3,5,7,9$ and 45. And all these are proved by Berndt and Chan [7] . Recently Yi [24], evaluated $\varphi(e^{-n\pi})$ for $n = 1,2,3,4,5$ and 6 and $\varphi(-e^{n\pi})$ for $n = 1,2,4,6,8,10$ and 12. Furthermore, M. S. M. Naika and Chandan kumar [13] and Naika et. al [14] established several general formulas for the explicit evaluation of $h_{2,n}$ and $h_{4,n}$ by employing modular equation of degree 2 and 4 respectively. On page 366 of his lost notebook [15, p. 248], Ramanujan recorded another continued fraction

$$G(q) := \frac{q^{1/3}}{1} {}_{+} \frac{q+q^2}{1} {}_{+} \frac{q^2+q^4}{1} {}_{+} \frac{q^3+q^6}{1} {}_{+\ldots}, \tag{1.3}$$

and claimed that there are many results of $G(q)$. Motivated by Ramanujan's claim, H. H. Chan [9] derived many new identities which Ramanujan vaguely referred. Recently B. C. Berndt, Chan and L. -C. Zhang [6], N. D. Baruah and Nipen Saikia [3], C. Adiga et. al. [1], S. Bhargava et. al. [8] have found several explicit values of $G(q)$. Further, as a particular case of this for $k = 3$ they proved the following:

Lemma 2. *For $q = e^{-\pi\sqrt{n/3}}$, let*

$$J_n := \frac{1}{\sqrt[4]{3}} \frac{\varphi(q)}{\varphi(q^3)},$$

then

i. $J_n J_{1/n} = 1$,

ii. $J_1 = 1$,

iii. $D_n = \sqrt{\dfrac{\sqrt{3} - J_n^2}{1+\sqrt{3}J_n^2}}$,

iv. $G(q) = \frac{1}{2}\sqrt[3]{1-3D_n^4}$.

Motivated by the above work, in this paper we find some general formulas for the explicit evaluation of $h_{2,n}$, $h_{3,n}$ and $h'_{3,n}$. Also we evaluate Ramanujan's cubic continued fraction and Ramanujan-Gӧllintz-Gordon continued fraction.

2. Preliminary results: P-Q type theta-function identities

In this section, we state P-Q type theta-function identities and also some $h_{k,n}$ and $h'_{k,n}$ which we need in sequel.

Theorem 1 ([8]). *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(-q)}{\varphi(-q^3)}$ *then*

$$\frac{P}{Q} + \frac{Q}{P} = \frac{3}{PQ} - PQ.$$

Theorem 2 ([8]). *If* $P := \dfrac{\varphi(-q)}{\varphi(-q^3)}$ *and* $Q := \dfrac{\varphi(-q^2)}{\varphi(-q^6)}$ *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 = \frac{3}{Q^2} - Q^2.$$

Theorem 3 ([8]). *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^2)}{\varphi(q^6)}$ *then*

$$(PQ)^2 + \frac{9}{(PQ)^2} + 2(P^2 - Q^2) = 6\left(\frac{1}{P^2} - \frac{1}{Q^2}\right) - \left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2 + 12.$$

Theorem 4 ([22]). *If* $P := \dfrac{\varphi^2(q^2)}{\varphi^2(q)}$ *and* $Q := \dfrac{\varphi^2(q^4)}{\varphi^2(q^2)}$ *then*

$$4P^2Q^2 - 4P^2Q + P^2 - 2P + 1 = 0.$$

Theorem 5 ([22]). *If* $P := \dfrac{\varphi^2(q^2)}{\varphi^2(q)}$ *and* $Q := \dfrac{\varphi^2(q^6)}{\varphi^2(q^3)}$ *then*

$$64PQ + \frac{16}{PQ} - 96(P+Q) - 48\left(\frac{1}{P} + \frac{1}{Q}\right) + 138 = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 36\left(\frac{P}{Q} + \frac{Q}{P}\right).$$

Theorem 6 ([20]). *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^7)}{\varphi(q^{21})}$ *then*

$$\left(\frac{P}{Q}\right)^4 - \left(\frac{Q}{P}\right)^4 + 14\left(\left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2\right) - 7\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - 1\right)$$
$$\times\left(PQ + \frac{3}{PQ}\right) + (PQ)^3 + \frac{27}{(PQ)^3} = 0.$$

Theorem 7 ([20]). *If* $P := \dfrac{\varphi(q^3)\varphi(q^2)}{\varphi(q)\varphi(q^6)}$ *and* $Q := \dfrac{\varphi(q^6)\varphi(q^4)}{\varphi(q^2)\varphi(q^{12})}$ *then*

$$(PQ)^2 + \frac{1}{(PQ)^2} - 8\left(PQ + \frac{1}{PQ}\right) + 6\left(\sqrt{PQ} - \frac{1}{\sqrt{PQ}}\right)\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right)$$
$$-2\left((PQ)^{3/2} - \frac{1}{(PQ)^{3/2}}\right)\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right) + \left(PQ + \frac{1}{PQ}\right)\left(\frac{P}{Q} + \frac{Q}{P}\right) + 10 = 0.$$

Theorem 8 ([20]). *If* $P := \dfrac{\varphi^2(q^3)}{\varphi(q)\varphi(q^9)}$ *and* $Q := \dfrac{\varphi^2(q^6)}{\varphi(q^2)\varphi(q^{18})}$ *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - 2\left(PQ - \frac{3}{PQ}\right)\left(\frac{P}{Q} + \frac{Q}{P}\right) + (PQ)^2 + \frac{9}{(PQ)^2}$$

$$-16\left(PQ+\frac{3}{PQ}\right)-44=0.$$

Theorem 9 ([20]). *If* $P:=\dfrac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})}$ *and* $Q:=\dfrac{\varphi(q^6)\varphi(q^{10})}{\varphi(q^2)\varphi(q^{30})}$ *then*

$$(PQ)^2+\frac{1}{(PQ)^2}-24\left(PQ+\frac{1}{PQ}\right)+\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2+\left[6\left(PQ+\frac{1}{PQ}\right)-8\right]$$
$$\times\left(\frac{P}{Q}+\frac{Q}{P}\right)-4\left(\sqrt{PQ}-\frac{1}{\sqrt{PQ}}\right)\left(\left(\frac{P}{Q}\right)^{3/2}+\left(\frac{Q}{P}\right)^{3/2}\right)$$
$$+\left[16\left(\sqrt{PQ}-\frac{1}{\sqrt{PQ}}\right)-4\left((PQ)^{3/2}-\frac{1}{(PQ)^{3/2}}\right)\right]\left(\sqrt{\frac{P}{Q}}+\sqrt{\frac{Q}{P}}\right)+36=0.$$

Lemma 3 ([24]). *If* $h_{k,n}$ *and* $h'_{k,n}$ *are as defined as in* (1.1) *and* (1.2) *then, we have*

$$h_{2,4}=\sqrt{2}+1-\sqrt{\sqrt{2}+1},\ h_{2,8}=\frac{\sqrt{2+\sqrt{2}}}{\sqrt[4]{2}+1},\ h_{3,3}=(2\sqrt{3}-3)^{1/4}=\frac{3^{1/8}\sqrt{\sqrt{3}-1}}{2^{1/4}},$$
$$h_{3,1/3}=\left(\frac{2\sqrt{3}+3}{3}\right)^{1/4}=\frac{\sqrt{\sqrt{3}+1}}{2^{1/4}3^{1/8}},\quad h_{3,5}=\frac{\sqrt{\sqrt{5}-1}}{\sqrt{2}},\quad h_{3,1/5}=\frac{\sqrt{\sqrt{5}+1}}{\sqrt{2}},$$
$$h_{3,9}=\frac{1}{\sqrt{3}}\left(1-\sqrt[3]{2}+\sqrt[3]{4}\right),\quad h_{3,1/9}=\frac{1+\sqrt[3]{2}}{\sqrt{3}},\quad h'_{3,1}=2^{-1/4}\sqrt{\sqrt{3}-1}.$$

3. Evaluation of $h_{k,n}$ and $h'_{k,n}$

Theorem 10. *We have*

i. $$h_{3,6}=\sqrt[4]{603-426\sqrt{2}-348\sqrt{3}+246\sqrt{6}}=h^{-1}_{3,1/6},$$

ii. $$h_{3,2/3}=\frac{1}{3}\sqrt[4]{603-426\sqrt{2}-348\sqrt{3}+246\sqrt{6}}\sqrt{9+6\sqrt{2}}=h^{-1}_{3,3/2}.$$

Proof. On using the definition of $h_{k,n}$ in Theorem 8 and by setting $n=1/6$, we deduce

$$x^2+\frac{9}{x^2}-16\left(x+\frac{3}{x}\right)-4\left(x-\frac{3}{x}\right)+46=0,$$

where $x=\left(\dfrac{h_{3,6}}{h_{3,2/3}}\right)^2$. Now set $\dfrac{x}{\sqrt{3}}+\dfrac{\sqrt{3}}{x}=t$, $\dfrac{x^2}{3}+\dfrac{3}{x^2}=t^2-2$ and $\dfrac{x}{\sqrt{3}}-\dfrac{\sqrt{3}}{x}=\sqrt{t^2-4}$ in the above, we obtain

$$9t^4-96\sqrt{3}t^3+960t^2-1280\sqrt{3}t+1792=0.$$

Since $h_{k,n}$ is decreasing, we choose $t = \frac{4}{\sqrt{3}}(3-\sqrt{2})$. Again on solving and since $\dfrac{h_{3,6}}{h_{3,2/3}} < 1$, we have

$$\frac{h_{3,6}}{h_{3,2/3}} = \sqrt{9-6\sqrt{2}}. \tag{3.1}$$

From [8, Theorem 3.4] , if $P := \dfrac{\varphi^2(q)}{\varphi^2(q^3)}$ and $Q := \dfrac{\varphi^2(q^3)}{\varphi^2(q^9)}$ then

$$PQ + \frac{9}{PQ} = 3 + 6\frac{Q}{P} + \frac{Q^2}{P^2}.$$

On employing the definition of $h_{k,n}$ in the above, setting $n = 1/6$ and using (3.1), we deduce

$$y^2 - (70-48\sqrt{2})y + 1 = 0,$$

where $y = (h_{3,6}h_{3,2/3})^2$. On solving, we choose

$$h_{3,6}h_{3,2/3} = \sqrt{35-24\sqrt{2}-20\sqrt{3}+14\sqrt{6}}.$$

From (3.1) and the above, we obtain the desired result. □

Corollary 1. *We have*

i. $$D_6 = \sqrt{\frac{\sqrt{3}-a}{1+\sqrt{3}a}}, \qquad D_{1/6} = \sqrt{\frac{\sqrt{3}a-1}{a+\sqrt{3}}},$$

ii. $$D_{2/3} = \frac{1}{\sqrt{3}}\sqrt{3-\sqrt{6}}, \qquad D_{3/2} = \sqrt{\frac{\sqrt{3}b^2-1}{b^2+\sqrt{3}}},$$

where

$$a = \sqrt{603-426\sqrt{2}-348\sqrt{3}+246\sqrt{6}} \qquad \textit{and} \qquad b = \frac{1}{3}\sqrt{a(9+6\sqrt{2})}.$$

Proof. On using Theorem 10 to Lemma 2(iii), we obtain the above results. □

Corollary 2. *We have*

i. $$G(e^{-\sqrt{2}\pi}) = \frac{1}{2}\sqrt[3]{1-3D_6^4}, \qquad G(e^{-\pi/3\sqrt{2}}) = \frac{1}{2}\sqrt[3]{1-3D_{1/6}^4},$$

ii. $$G(e^{-\sqrt{2}\pi/3}) = \frac{1}{2}\sqrt[3]{1-3D_{2/3}^4}, \qquad G(e^{-\pi/\sqrt{2}}) = \frac{1}{2}\sqrt[3]{1-3D_{3/2}^4}.$$

Proof. To prove the above results, we need to apply Corollary 1 to Lemma 2(iv) respectively. □

Theorem 11. *We have*

i. $$h_{3,10} = \sqrt{(1+\sqrt{2})(2-\sqrt{3})}\sqrt[4]{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}} = h^{-1}_{3,1/10},$$

ii. $$h_{3,2/5} = \frac{\sqrt[4]{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}}}{\sqrt{(1+\sqrt{2})(2-\sqrt{3})}} = h_{3,5/2}^{-1}.$$

Proof. On using the definition of $h_{k,n}$ in Theorem 9 and by setting $n = 1/10$, we deduce

$$x^4 + \frac{1}{x^4} - 8\left(x^3 - \frac{1}{x^3}\right) - 12\left(x^2 + \frac{1}{x^2}\right) + 24\left(x - \frac{1}{x}\right) + 22 = 0,$$

where $x = \dfrac{h_{3,10}}{h_{3,2/5}}$. Setting $x - x^{-1} = t$ in the above, we obtain $t^2\left(t^2 - 8t - 8\right) = 0$. Since $h_{k,n}$ is positive and decreasing, we have $x - \dfrac{1}{x} = 4 - 2\sqrt{6}$. On solving this, we deduce

$$\frac{h_{3,10}}{h_{3,2/5}} = (1+\sqrt{2})(2-\sqrt{3}). \tag{3.2}$$

From [21, Theorem 3.1], if $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ and $Q := \dfrac{\varphi(q^5)}{\varphi(q^{15})}$ then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 + 5\left(\left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2\right) + 5\left(\frac{Q}{P} - \frac{P}{Q}\right).$$

Again using the definition of $h_{k,n}$ in the above, by setting $n = 2/5$ and then using (3.2), we deduce

$$x^2 + \frac{1}{x^2} = 198 - 80\sqrt{6},$$

where $x = (h_{3,2/5}h_{3,10})^2$. On solving this, we obtain

$$h_{3,2/5}h_{3,10} = \sqrt{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}}.$$

On using (3.2) and the above, we obtain the desired result. □

Corollary 3. *We have*

i. $$D_{10} = \sqrt{\frac{\sqrt{3}-ab}{1+\sqrt{3}ab}}, \qquad D_{1/10} = \sqrt{\frac{\sqrt{3}ab-1}{ab+\sqrt{3}}},$$

ii. $$D_{2/5} = \sqrt{\frac{1+\sqrt{2}-a-2b}{b-\sqrt{3}a}}, \qquad D_{5/2} = \sqrt{\frac{\sqrt{3}a-b}{a+\sqrt{3}b}},$$

where

$$a = \sqrt{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}} \qquad \textit{and} \qquad b = 2-\sqrt{3}+2\sqrt{2}-\sqrt{6}.$$

Proof. On using Theorem 11 to Lemma 2(iii), we obtain the above results. □

Corollary 4. *We have*

i. $G(e^{-\pi\sqrt{10/3}}) = \frac{1}{2}\sqrt[3]{1-3D_{10}^4}, \qquad G(e^{-\pi\sqrt{1/30}}) = \frac{1}{2}\sqrt[3]{1-3D_{1/10}^4},$

ii. $G(e^{-\pi\sqrt{2/15}}) = \frac{1}{2}\sqrt[3]{1-3D_{2/5}^4}, \qquad G(e^{-\pi\sqrt{5/6}}) = \frac{1}{2}\sqrt[3]{1-3D_{5/2}^4}.$

Proof. To prove the above results, we need to apply Corollary 3 to Lemma $2(iv)$ respectively. □

Theorem 12. *We have*

i. $h_{2,3} = \sqrt{\sqrt{6}+\sqrt{3}-\sqrt{2}-2} = h_{2,1/3}^{-1},$

ii. $h_{2,1/9} = \sqrt{35-24\sqrt{2}-2\sqrt{6(99-70\sqrt{2})}} = h_{2,9}^{-1}.$

Proof. On employing the definition of $h_{k,n}$ in Theorem 5 and setting $n = 1/3$, we deduce

$$x^4+\frac{1}{x^4}-36\left(x^2+\frac{1}{x^2}\right)+96\sqrt{2}\left(x+\frac{1}{x}\right)+202=0.$$

where $x = h_{2,3}^2$. On setting $x+x^{-1} = t$, we obtain

$$t^4-40t^2+96\sqrt{2}t-128=0.$$

On solving, we obtain $t = -2(\sqrt{6}+\sqrt{2})$, $2(\sqrt{6}-\sqrt{2})$ and $2\sqrt{2}$ as a double root. Since $h_{k,n}$ is a decreasing function, we choose

$$x+\frac{1}{x} = 2(\sqrt{6}-\sqrt{2}).$$

On solving this, we obtain the first identity. Similarly we obtain $h_{2,1/9}$ by setting $n = 1/9$ in Theorem 5. □

Corollary 5. *We have*

i. $H(e^{-\pi\sqrt{3/2}}) = \sqrt{\frac{\sqrt[4]{2}h_{2,3}-1}{\sqrt[4]{2}h_{2,3}+1}}, \qquad H(e^{-\pi/\sqrt{6}}) = \sqrt{\frac{\sqrt[4]{2}-h_{2,1/3}}{\sqrt[4]{2}+h_{2,1/3}}},$

ii. $H(e^{-\pi/3\sqrt{2}}) = \sqrt{\frac{\sqrt[4]{2}h_{2,1/9}-1}{\sqrt[4]{2}h_{2,1/9}+1}}, \qquad H(e^{-3\pi/\sqrt{2}}) = \sqrt{\frac{\sqrt[4]{2}-h_{2,9}}{\sqrt[4]{2}+h_{2,9}}}.$

Proof. The above results directly follows from Lemma 1 and Theorem 12. □

Theorem 13. *We have*

$$h_{3,4} = \frac{1}{2}\left(2-\sqrt{6}+\sqrt{14-4\sqrt{6}}\right) = h_{3,1/4}^{-1}.$$

Proof. On using the definition of $h_{k,n}$ in Theorem 7 and setting $n = 1/4$, we deduce

$$x^4+\frac{1}{x^4}-4\left(x^3-\frac{1}{x^3}\right)-6\left(x^2+\frac{1}{x^2}\right)+12\left(x-\frac{1}{x}\right)+10=0,$$

where $x = h_{3,4}^2$ and on setting $x + x^{-1} = t$, we have

$$t^2(t^2 - 4t - 2) = 0.$$

On solving, we obtain $t = 0$, $2 + \sqrt{6}$ and $2 - \sqrt{6}$. Since $0 < h_{k,n} < 1$, we choose $x - x^{-1} = 2 - \sqrt{6}$. Further on solving this, we obtain $h_{3,4}$. □

Corollary 6. *We have*

$$D_4 = \sqrt{\frac{4\sqrt{2} + 2\sqrt{6} - 2\sqrt{3} - 6}{6\sqrt{3} - 6\sqrt{2} - 4\sqrt{6} + 10}}, \quad D_{1/4} = \sqrt{\frac{6\sqrt{3} - 6\sqrt{2} - 4\sqrt{6} + 8}{4\sqrt{3} - 2\sqrt{6} - 4\sqrt{2} + 6}}.$$

Proof. On using Theorem 13 to Lemma 2(iii), we obtain D_4 and $D_{1/4}$. □

Corollary 7. *We have*

$$G(e^{-2\pi/\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1 - 3D_4^4}, \qquad G(e^{-\pi/2\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1 - 3D_{1/4}^4}.$$

Proof. To prove the above results, we need to apply Corollary 6 to Lemma 2(iv) respectively. □

Theorem 14. *We have*

$$h_{3,7} = \left(2\sqrt{7\left(7 + 4\sqrt{3}\right)} - 6\sqrt{3} - 9\right)^{1/4} = h_{3,1/7}^{-1}.$$

Proof. On using the definition of $h_{k,n}$ in Theorem 6 and setting $n = 1/7$, we deduce

$$\frac{1}{x^4} - x^4 + 14\left(\frac{1}{x^2} - x^2\right) - 14\sqrt{3}\left(\frac{1}{x^2} + x^2\right) + 20\sqrt{3} = 0,$$

where $x = h_{3,7}^2$ and on solving we obtain the above result. □

Corollary 8. *We have*

$$D_7 = \sqrt{\frac{\sqrt{3} - \sqrt{a}}{1 + \sqrt{3a}}}, \qquad D_{1/7} = \sqrt{\frac{\sqrt{3}a - 1}{a + \sqrt{3}}},$$

where

$$a = 2\sqrt{7(7 + 4\sqrt{3})} - 6\sqrt{3} - 9.$$

Proof. On using Theorem 14 to Lemma 2(iii), we obtain D_7 and $D_{1/7}$. □

Corollary 9. *We have*

$$G(e^{-\pi\sqrt{7/3}}) = \frac{1}{2}\sqrt[3]{1 - 3D_7^4}, \qquad G(e^{-\pi/\sqrt{21}}) = \frac{1}{2}\sqrt[3]{1 - 3D_{1/7}^4}.$$

Proof. To prove the above results, we need to apply Corollary 8 to Lemma 2(iv) respectively. □

Theorem 15. *We have*

i. $h_{3,2} = \sqrt{\sqrt{3}+\sqrt{8-4\sqrt{3}}-2} = h^{-1}_{3,1/2}$,

ii. $h_{3,12} = \sqrt{\dfrac{9+3\sqrt[4]{3}\sqrt{2}-3\sqrt{3}}{A+B}} = h^{-1}_{3,1/12}$,

iii. $h_{3,4/3} = \sqrt{\dfrac{3-\sqrt{2}\sqrt[4]{3}+3\sqrt{3}-2D}{C}} = h^{-1}_{3,3/4}$,

iv. $h_{3,20} = \sqrt{\dfrac{2\sqrt{15}+2\sqrt{3}+\sqrt{5}-9}{E}} = h^{-1}_{3,1/20}$,

v. $h_{3,4/5} = \sqrt{\dfrac{\sqrt{3}-6\sqrt{5}+\sqrt{15}+4G-6}{F}} = h^{-1}_{3,5/4}$,

vi. $h_{3,36} = \dfrac{1}{3}(H-I) = h^{-1}_{3,1/36}$,

vii. $h_{3,4/9} = \frac{1}{3}\left(3-\sqrt{3}+\sqrt[3]{4}\sqrt{3}+J\right) = h^{-1}_{3,9/4}$,

where

$$A = 9\sqrt{3}+6\sqrt{2}\sqrt[4]{3}-3\sqrt{2}\sqrt[4]{27}-9,$$

$$B = \sqrt{3\left(96-120\sqrt{2}\sqrt[4]{3}-48\sqrt{3}+72\sqrt{2}\sqrt[4]{27}\right)},$$

$$C = \sqrt{2}\sqrt[4]{27}+2\sqrt{2}\sqrt[4]{3}-\sqrt{3}-3, \qquad D = \sqrt{2\left(4-\sqrt{2}\sqrt[4]{3}+2\sqrt{3}-\sqrt{2}\sqrt[4]{27}\right)},$$

$$E = 6+\sqrt{3}-6\sqrt{5}-\sqrt{15}-4\sqrt{2(2+\sqrt{3})(3-\sqrt{5})},$$

$$F = 2\sqrt{15}-3\sqrt{5}+2\sqrt{3}-11, \qquad G = \sqrt{2(2-\sqrt{3})(3+\sqrt{5})},$$

$$H = 45+36\sqrt[3]{2}+27\sqrt[3]{4}+(25+20\sqrt[3]{2}+16\sqrt[3]{4})\sqrt{3},$$

$$I = \sqrt{11430+9072\sqrt[3]{2}+7200\sqrt[3]{4}+6597\sqrt{3}+5238\sqrt[3]{2}\sqrt{3}+4158\sqrt[3]{4}\sqrt{3}},$$

$$J = \sqrt{6+12\sqrt[3]{2}-12\sqrt[3]{4}-3\sqrt{3}-6\sqrt[3]{2}\sqrt{3}+6\sqrt[3]{4}\sqrt{3}}.$$

Proof. The above results directly follows from Theorem 3 and the definition of $h_{k,n}$, where we set $n = 1/2, 1, 3, 1/3, 5, 1/5, 9$ and $1/9$ respectively by making use of Lemma 3. □

Corollary 10. *We have*

i. $D_2 = \sqrt{\sqrt{2}-1}$, $\qquad D_{1/2} = \sqrt{\dfrac{2-2\sqrt{3}+\sqrt{3(8-4\sqrt{3})}}{2\sqrt{3}-2+\sqrt{8-4\sqrt{3}}}}$,

ii. $D_{12} = \sqrt{\dfrac{\sqrt{3}-a^2}{1+\sqrt{3}a^2}}$, $\qquad D_{1/12} = \sqrt{\dfrac{\sqrt{3}a^2-1}{a^2+\sqrt{3}}}$,

iii. $D_{4/3}=\sqrt{\dfrac{\sqrt{3}-b^2}{1+\sqrt{3}b^2}}$, $\quad D_{3/4}=\sqrt{\dfrac{\sqrt{3}b^2-1}{b^2+\sqrt{3}}}$,

iv. $D_{20}=\sqrt{\dfrac{\sqrt{3}-c^2}{1+\sqrt{3}c^2}}$, $\quad D_{1/20}=\sqrt{\dfrac{\sqrt{3}c^2-1}{c^2+\sqrt{3}}}$,

v. $D_{4/5}=\sqrt{\dfrac{\sqrt{3}-d^2}{1+\sqrt{3}d^2}}$, $\quad D_{5/4}=\sqrt{\dfrac{\sqrt{3}d^2-1}{d^2+\sqrt{3}}}$,

vi. $D_{36}=\sqrt{\dfrac{\sqrt{3}-e^2}{1+\sqrt{3}e^2}}$, $\quad D_{1/36}=\sqrt{\dfrac{\sqrt{3}e^2-1}{e^2+\sqrt{3}}}$,

vii. $D_{4/9}=\sqrt{\dfrac{\sqrt{3}-f^2}{1+\sqrt{3}f^2}}$, $\quad D_{9/4}=\sqrt{\dfrac{\sqrt{3}f^2-1}{f^2+\sqrt{3}}}$,

where

$$a=\sqrt{\frac{9+3\sqrt[4]{3}\sqrt{2}-3\sqrt{3}}{A+B}}, \qquad b=\sqrt{\frac{3-\sqrt{2}\sqrt[4]{3}+3\sqrt{3}-2D}{C}},$$

$$c=\sqrt{\frac{2\sqrt{15}+2\sqrt{3}+\sqrt{5}-9}{E}}, \qquad d=\sqrt{\frac{\sqrt{3}-6\sqrt{5}+\sqrt{15}+4G-6}{F}},$$

$$e=\frac{1}{3}(H-I), \qquad f=\frac{1}{3}\left(3-\sqrt{3}+\sqrt[3]{4}\sqrt{3}+J\right).$$

Here A, B, C, D, E, F, G, H, Iand J are as defined in Theorem 15.

Proof. On using Theorem 15 to Lemma $2(iii)$, we obtain the above results. □

Corollary 11. *We have*

i. $G(e^{-\pi\sqrt{2/3}})=\dfrac{1}{2}\sqrt[3]{36\sqrt{2}-50}$, $\quad G(e^{-\pi/\sqrt{6}})=\dfrac{1}{2}\sqrt[3]{1-3D_{1/2}^4}$,

ii. $G(e^{-2\pi})=\dfrac{1}{2}\sqrt[3]{1-3D_{12}^4}$, $\quad G(e^{-\pi/6})=\dfrac{1}{2}\sqrt[3]{1-3D_{1/12}^4}$,

iii. $G(e^{-2\pi/3})=\dfrac{1}{2}\sqrt[3]{1-3D_{4/3}^4}$, $\quad G(e^{-\pi/2})=\dfrac{1}{2}\sqrt[3]{1-3D_{3/4}^4}$,

iv. $G(e^{-\pi\sqrt{20/3}})=\dfrac{1}{2}\sqrt[3]{1-3D_{20}^4}$, $\quad G(e^{-\pi 2\sqrt{15}})=\dfrac{1}{2}\sqrt[3]{1-3D_{1/20}^4}$,

v. $G(e^{-2\pi/\sqrt{15}})=\dfrac{1}{2}\sqrt[3]{1-3D_{4/5}^4}$, $\quad G(e^{-\sqrt{5}\pi/2\sqrt{3}})=\dfrac{1}{2}\sqrt[3]{1-3D_{5/4}^4}$,

vi. $G(e^{-2\sqrt{3}\pi})=\dfrac{1}{2}\sqrt[3]{1-3D_{36}^4}$, $\quad G(e^{-\pi/6\sqrt{3}})=\dfrac{1}{2}\sqrt[3]{1-3D_{1/36}^4}$,

vii. $G(e^{-2\pi/3\sqrt{3}})=\dfrac{1}{2}\sqrt[3]{1-3D_{4/9}^4}$, $\quad G(e^{-\sqrt{3}\pi/2})=\dfrac{1}{2}\sqrt[3]{1-3D_{9/4}^4}$.

Proof. To prove the above results, we need to apply Corollary 10 to Lemma $2(iv)$ respectively. □

Theorem 16. *We have*

$$h'_{3,1/3}=\sqrt{\frac{6\sqrt{3}-\sqrt{2}\sqrt[4]{27}-3\sqrt{2}\sqrt[4]{3}}{3(2+\sqrt{2}\sqrt[4]{3}+\sqrt{2}\sqrt[4]{27})}},\ h'_{3,5}=\sqrt{\frac{\sqrt{6\left(\sqrt{5}+1\right)}-\sqrt{2\left(\sqrt{5}-1\right)}}{\sqrt{2\left(\sqrt{5}+1\right)}+\sqrt{6\left(\sqrt{5}-1\right)}}},$$

$$h'_{3,1/5}=\sqrt{\frac{\sqrt{6\left(\sqrt{5}-1\right)}-\sqrt{2\left(\sqrt{5}+1\right)}}{\sqrt{2\left(\sqrt{5}-1\right)}+\sqrt{6\left(\sqrt{5}+1\right)}}},\qquad h'_{3,9}=\sqrt{\frac{\sqrt{3}-\sqrt[3]{4}\sqrt{3}+3}{\sqrt{3}+3\sqrt[3]{4}-3}},$$

$$h'_{3,1/9}=\sqrt{\frac{9-\sqrt{3}-2\sqrt[3]{2}\sqrt{3}-\sqrt[3]{4}\sqrt{3}}{3(1+2\sqrt[3]{2}+\sqrt[3]{4}+\sqrt{3})}}.$$

Proof. The above results directly follows from Theorem 1 and the definition of $h_{k,n}$, where we set $n=1/3,5,1/5,9$ and $1/9$ respectively by making use of Lemma 3. □

Theorem 17. *We have*

i. $$h_{2,16}=\frac{2}{\sqrt{2\sqrt{2}+4\sqrt{2\left((13\sqrt{2}+18)\sqrt{1+\sqrt{2}}-20\sqrt{2}-28\right)}}}=h^{-1}_{2,1/16},$$

ii. $$h_{2,32}=\sqrt{\frac{6+8\sqrt[4]{2}+12\sqrt{2}+8\sqrt[4]{8}}{A}}=h^{-1}_{2,1/32},$$

where

$$A=12+8\sqrt[4]{2}+3\sqrt{2}+4\sqrt[4]{8}+\sqrt{256+240\sqrt[4]{2}+192\sqrt{2}+144\sqrt[4]{8}}.$$

Proof. The above results directly follows from Theorem 4 and the definition of $h_{k,n}$, where we set $n=4$ and 16 respectively by making use of Lemma 3. □

Corollary 12. *We have*

i. $$H(e^{-2\sqrt{2}\pi})=\sqrt{\frac{\sqrt[4]{2}h_{2,16}-1}{\sqrt[4]{2}h_{2,16}+1}},\qquad H(e^{-\pi/4\sqrt{2}})=\sqrt{\frac{\sqrt[4]{2}-h_{2,1/16}}{\sqrt[4]{2}+h_{2,1/16}}},$$

ii. $$H(e^{-4\pi})=\sqrt{\frac{\sqrt[4]{2}h_{2,32}-1}{\sqrt[4]{2}h_{2,32}+1}},\qquad H(e^{-\pi/8})=\sqrt{\frac{\sqrt[4]{2}-h_{2,1/32}}{\sqrt[4]{2}+h_{2,1/32}}}.$$

Proof. The above results directly follows from Lemma 1 and Theorem 17. □

Theorem 18. *We have*

$$h'_{3,4}=\frac{\sqrt{3}-1}{\sqrt{2}},\qquad \textit{and}\qquad h'_{3,1/4}=\left(\frac{3-2\sqrt{2}-\sqrt{3}+\sqrt{6}}{3+\sqrt{2}-\sqrt{3}}\right)^{1/4}.$$

Proof. The above results directly follows from Theorem 2 and the definition of $h_{k,n}$, where we set $n = 1$ and $1/4$ respectively by making use of Lemma 3. □

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